

The non-existence of mixed-strategy equilibria in multi-dimensional all-pay auctions*

J. Atsu Amegashie

Department of Economics
Simon Fraser University
Burnaby, British Columbia
Canada V5A 1S6

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E-mail: jamegash@sfu.ca
Tel: (604) 298-0467
Fax: (604) 291-5944

Abstract

I show that there is no mixed-strategy equilibrium in multi-dimensional all-pay auctions, if no caps are placed on the bids of the contestants.

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1. Introduction and analysis

The all-pay auction has been used to examine many allocation processes and interesting economic phenomena. The analysis has focussed on all-pay auctions in which contestants compete on a single dimension. But in practice all-pay auctions could have multiple dimensions. For example, applicants for jobs are evaluated on several dimensions (i.e., analytical skills, presentation skills, etc). Alternatively, there could be a single dimension but the outcome of the hiring process may be determined by a committee (not a single administrator). If the contestants have to impress or lobby each member of the committee to get his vote, then the number of committee members is analytically equivalent to the number of dimensions.

It is a standard result that an all-pay auction with a single dimension and a single administrator has no equilibrium in pure-strategies but has an equilibrium in mixed strategies in which the players randomize their bids over some finite support, with or without a cap on the bids of contestants [Baye, Kovenock, and de Vries (1996), Che and Gale (1998), and Hillman and Riley (1989)]. In this note, I show that there is no mixed-strategy equilibrium in multi-dimensional all-pay auctions, if no caps are placed on the bids of the contestants.¹ For the sake of exposition, I focus on the case of a single dimension with multiple administrators (i.e., a committee). As noted above, this is analytically equivalent to the case of a single administrator with multiple dimensions.

Without loss of generality but for the sake of exposition, consider two identical and risk-neutral agents (1 and 2) contesting for a prize commonly known to be valued at

¹ The proof for the non-existence of a pure-strategy equilibrium is very similar to the proof for the case of a single dimension.

$V > 0$. Suppose that the prize will be awarded by a committee of two members (call them A and B). Suppose the contest is an all-pay auction. In an all-pay auction, the contestant with the highest bid wins with certainty and ties are broken randomly; losers forfeit their bids. A contestant wins if both committee members vote for him. This occurs if he spends (bids) more on each of the two committee members than the other contestant. If the contestants spend the same amount on a given committee member, then that member votes randomly for one of the contestants. If the committee members do not vote for the same contestant, then a random mechanism is used to pick the winner. In this contest, the prize will necessarily be awarded.

Suppose there are no caps on the bids of the contestants. Let V be the upper bound of player 1's bids on committee member A in a mixed-strategy equilibrium. Then when player 1 bids V on A, he must bid zero on B since it does not make sense for his total bid to exceed his valuation. Then he will win the prize with probability $\beta < 1$, since we know that committee member B might *not* vote for him. His expected payoff is $\beta V - V < 0$. Hence player 1 will never bid V on A. Therefore V cannot be the upper bound of player 1's bids in a mixed-strategy equilibrium.

Suppose now that the upper bound of player 1's bid is $\hat{V} < V$ (for committee member A) and suppose player 2 is considering bidding \hat{V} on A and b_2^B on B, where $\hat{V} + b_2^B < V$. Then player 2 is better off if he bids *marginally higher* than \hat{V} on A instead of bidding on some interval $[\hat{V} - \epsilon, \hat{V}]$ since his probability of success rises *discontinuously* by doing so, where $\epsilon > 0$. Hence there is some interval $[\hat{V} - \epsilon, \hat{V}]$ over which player 2 will not bid on A. But then \hat{V} cannot be the upper bound of player 1's

bids on A since he can move the upper bound of his bids below \hat{V} without changing his probability of winning. Now consider $\hat{V} + b_2^B = V$. I want to show that it is profitable for player 2 to deviate by bidding marginally higher than \hat{V} on A and bidding less than b_2^B on B. Suppose that player 1's equilibrium mixed strategy on B is given by the distribution function $F_1^B(x)$ which gives the probability that player 1 bids less than x on B. Suppose $F_1^B(x)$ is continuous on some interval $[b_2^B - \epsilon', b_2^B]$, where $\epsilon' > 0$. Then player 2's success probability rises *discontinuously* when he bids *marginally* more than \hat{V} on A and *marginally* less than b_2^B on B, given that $F_1^B(x)$ is *continuous* on the interval $[b_2^B - \epsilon', b_2^B]$ and \hat{V} is the upper bound of the support of player 1's mixed-strategy equilibrium bids on A. Suppose instead that $F_1^B(x)$ is discontinuous at b_2^B such that player 1 never bids on some interval $[b_2^B - \epsilon'', b_2^B]$, where $\epsilon'' > 0$. Then player 2's success probability rises discontinuously when he bids marginally more than \hat{V} on A and marginally less than b_2^B on B. Hence there is some interval $[\hat{V} - \epsilon, \hat{V}]$ over which player 2 will not bid on A. But then \hat{V} cannot be the upper bound of player 1's bid on A since he can move the upper bound of his bids below \hat{V} without changing his probability of winning. Thus, there is no mixed-strategy equilibrium in which the players randomize on some finite support if no caps are placed on the bids of contestants. Thus I state the following proposition:

Proposition: *There is no mixed-strategy equilibrium in multi-dimensional all-pay auctions, if a cap is not placed on the bids of the contestants.*

The proposition above is useful because it saves the research effort of looking for a solution to a problem which has no solution. Indeed, placing a cap on the bids of contestants in multi-dimensional all-pay auctions could provide a solution [see Amegashie (2001), and Van Cayseele et. al (1993)].² It is important to note that placing a cap on the bids is not an artificial restriction on the game since there are caps on lobbying in the real world (Che and Gale, 1998).

Theil (1988) shows how a multi-dimensional procurement problem can be translated into a one-dimensional problem if the seller's objective function is known. Placing a cap on the bids of the contestants could transform a multi-dimensional all-pay auction into a single dimensional all-pay auction. To see this, consider the two-member committee case with two players. Each committee member could be viewed as a single administrator awarding a prize of $V/2$. That is, there are two separate single dimensional all-pay auctions each with a prize of $V/2$. It is known that in a single dimensional all-pay auction with a single administrator and two contestants, the contestants randomize their bids uniformly on $[0, V]$. Now place a cap of $V/2$ on the bids of the contestants in the two-player, two committee-member case. Then we can apply the result from the single dimensional case with a single administrator to show that in the two-member committee case, each contestant will bid on each of the committee members by randomizing uniformly on $[0, V/2]$.³ Note, however, that this transformation is not always possible.

² Myerson (1993) considers a similar model in which multiple candidates contest for political office. Each member of the electorate votes for the candidate who offers him/her a higher payoff based on the candidate's campaign promise. However, Myerson's model is not an all-pay auction because the candidates do not incur the cost of lobbying; the campaign promises are not costly to the contestants.

³ See Amegashie (2001).

For example, it does not work with two committee members who determine the winner through a unanimity rule. That is, the prize is not awarded if they vote for different contestants.

It is hoped this note will stimulate further research on multi-dimensional all-pay auctions given that there is very little work on the subject.

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